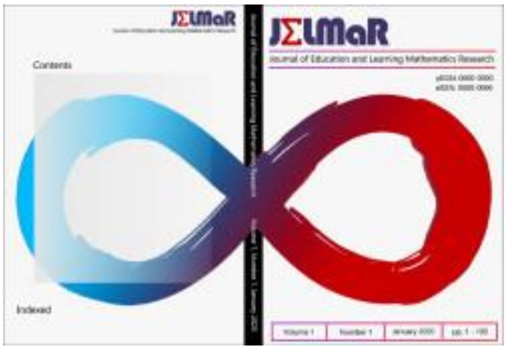

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Proof -Based Learning Design Using APOS Theory on Triangle Inequality Theorem Material

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***Abstract**: This research aims to produce a series of activities that can support evidence-based learning on the material of the Triangle Inequality Theorem. The research method used was design research type validation study and involved 22 4th semester students of the Mathematics Education study program at PGRI University Palembang. The research was conducted through 3 stages, namely preliminary design, design experiment and restropective analysis. Data collection was carried out using observation techniques, documentation and activity sheets. The research data were analyzed qualitatively, namely comparing the results of observations during the learning process with the HLT that had been designed. The results showed that the designed activities can support evidence-based learning for Triangle Inequality Theorem material.*

Keyword: learning trajectory, proof, triangle inequality theorem, APOS theory

INTRODUCTION

Real analysis is the basis in mathematics to think formally, namely thinking axiomatically deductive (Kartika & Yazidah, 2019). Real analysis has quite abstract characteristics and contains quite a lot of theorems. One of the important theorems that uses the concept of absolute value of real numbers and is used in various branches of mathematics is the triangle inequality theorem. In real life, civil engineers use the triangle inequality theorem because their line of work deals with surveying, transportation, and urban planning. The triangle inequality theorem helps them calculate unknown lengths and have rough estimates of various dimensions.

The existing theorem must be proven. According to (Gunawan, 2009) proof is something that distinguishes mathematics from other sciences such as physics or chemistry that is based on experiments. In mathematics, experiments are also important but proof is more essential. In doing proof, accuracy, skill, and precision are needed in understanding the meaning of the theorem so that it can be proven correct. According to Reid (2001)) proof is basically making a series of deductions from assumptions (premises or axioms) and existing mathematical results (lemmas or theorems) to obtain important results from a mathematical problem. Therefore, in the mathematics curriculum, proof acts as the main key to demonstrate mathematical understanding (Hanna, 2000).

In fact, the ability to do proof is often a major obstacle for students. Understanding formal definitions to proofs and their properties is a challenge for students (Darmadi, 2009). Students' misconceptions about the material often make it difficult to prove (Isnani, Waluya, Rochmad, Dwijantoe, & Asih, 2021). Therefore, a learning that is based on evidence is needed. (Lameena., Nusantara, & Muksar, 2018) Proof should be part of prospective teachers' mathematical experience in teacher education, because teachers' knowledge and beliefs about proof shape their readiness, willingness and capacity to support students' engagement in proof (Lameena., Nusantara, & Muksar, 2018).

One of the theories of mathematics learning that is very relevant to evidence-based learning is the APOS theory (Action, Process, Object, Schema). According to this theory, when someone tries to understand a mathematical idea, the process will start from a mental action towards the mathematical idea, and will eventually arrive at the construction of a scheme about certain mathematical concepts covered in the given problem (Dubinsky & McDonald, 2001). Some appropriate learning activities should be designed to support the construction of these mental categories (Syamsuri & Marethi, 2018). Through activities, students are invited to understand concepts and see the meaning contained in the material studied and its relationship to everyday life. The active involvement of students in the learning process will result in the natural cognitive development of students (Slavin R. , 2008).

Research results show that the use of APOS theory in evidence-based learning provides better learning outcomes (Erawati, 2018) (Borji, Alamolhodaei, & Radmehr, 2018) (Arnawa, Yanita, Ginting, Yerizon, & Nita, 2020), fosters positive attitudes towards mathematics (Arnawa, Yanita, Ginting, Yerizon, & Nita, 2020) and can improve student learning motivation and the quality of lecturer teaching (Erawati, 2018). However, there is little research that attempts to link classroom instruction activities and student learning (Melhuish, Fukawa-Connelly, Dawkins, Woods, & Weber, 2022). Thus, it is necessary to design learning activities based on APOS theory that can support evidence-based learning, especially on triangle inequality material.

METHOD

This research uses design research method, which is a systematic study of designing, developing and evaluating educational interventions (such as programs, learning strategies and materials, products and systems) as a solution to solve complex problems in educational practice, which also aims to advance knowledge about the characteristics of these interventions as well as the design and development process (Plomp, 2013). The research subjects were 22 students of the Mathematics Education study program at PGRI University Palembang participating in the Real Analysis course.

The data collection techniques used in this study are as follows: (1) Observation, which is direct observation made by researchers during the learning process that has been previously designed; (2) Documentation, in the form of video recordings, photos and sound recordings to see the activities of research subjects, interactions between research subjects and subjects with course lecturers; (3) Tests, in the form of student activity sheets to see the thinking process of students in the learning process. The research data were analyzed qualitatively, comparing the observation results during the learning process with the HLT that had been designed.

The research was conducted in 3 stages: preliminary design, design experiment and restropective analysis. The main purpose of the preliminary design stage is to develop a sequence of learning activities and design instruments to evaluate the learning process. How students think through various activities to achieve learning objectives is depicted in a learning trajectory (Rangkuti & Siregar, 2019). According to (Prahmana, 2017), Hypothetical Learning Trajectory (HLT) is a learning design and hypothesizes how students' understanding or presumed answers develop in learning activities. To make the HLT, a review of relevant literature, discussions with experienced peers and discussions with experts were first conducted. Furthermore, the HLT, learning tools and instruments used in the research were validated to experts.

After it was declared valid, it proceeded to the design experiment stage. At this stage, the HLT was elaborated and the teaching experiment revised. Literature exploration and research were conducted during this time. In addition, the first design was piloted to see how the learning activity plan went. The design trial was conducted in stages. The first stage was the pilot experiment. The HLT design was tested through 2 activities that were included in 2 meetings. During the pilot, the learning activities were observed by three observers. The observers were tasked with observing the implementation of learning using the HLT design. From this activity, inputs were obtained that might replace and revise the HLT activities. The second stage was the teaching experiment, which was conducted in a large group, namely the class attended by Real Analysis course participants. The revision of HLT into Learning Trajectory (LT) of evidence-based learning using APOS Theory was carried out at this stage. When the learning that is done is not in accordance with the design that has been designed, it is necessary to redesign (thought experiment) of the HLT and then test the HLT again (instruction experiment).

After the design experiment, Retrospective Analysis is conducted. Whatever happens in the classroom (seen from video recordings and observation sheets) is analyzed based on the design objectives so that it can be archived or not. At this stage, researchers analyzed the learning process at the teaching experiment stage (design experiment). The data analysis process was conducted by comparing the observation results during the learning process with the HLT. In the retrospective analysis, it was investigated and explained how students proved the Triangle Inequality Theorem.

RESULT AND DISCUSSION

The results obtained at each stage of the research procedure are described as follows:

1.Preliminary Design

Based on the analysis of the curriculum, syllabus and materials in the Mathematics Education study program at PGRI University Palembang, it is known that the Real Analysis course is given to 4th semester students with course learning outcomes, namely (i) Students can understand the language structure inherent in writing mathematical statements; (ii) Students are able to interpret the symbol system in mathematical writing through discussion of definitions, theorems, and accompanying proofs; (iii) Students can understand the meaning of the real number system, definitions and related theorems and are able to apply them in solving problems. The Course Learning Sub-Achievement is to understand the absolute value of real numbers and related theorems. The indicators are proving the properties of absolute value, proving triangle inequality, applying definitions and theorems to prove/solve absolute value problems and applying definitions and theorems to prove/solve triangle inequality problems.

To achieve what is targeted in the curriculum and especially the course syllabus, the HLT needs to be designed in such a way that what is predicted in the initial HLT does not deviate too far from the actual learning trajectory. HLT consists of three components that support each other, namely learning objectives, learning activities and learning tools or media used in the learning process as well as learning process conjectures (Prahmana, 2017). The HLT design for triangle inequality material is presented in Figure 1.

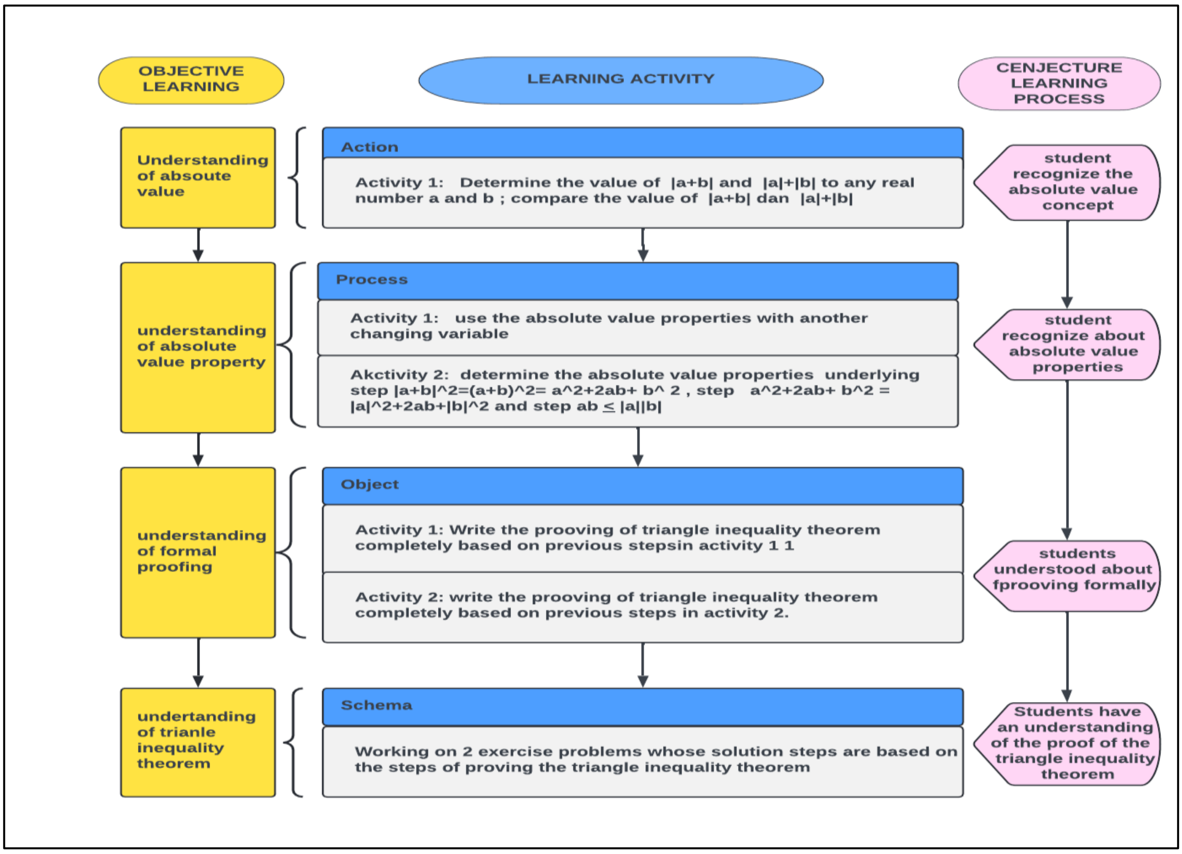


Figure 1: HLT for Proving Triangle Inequality

At this stage, the Student Activity Sheet (LAM) was also produced. Student worksheets or student activity sheets are printed teaching materials in the form of sheets of paper containing material, summaries, and instructions for implementing learning tasks that must be done by students, both theoretical and / or practical which refer to the competencies to be achieved by students, and their use depends on other teaching materials (Prastowo, 2015). LAM consists of 2 activities, where each activity consists of several questions / instructions as an implementation of APOS Theory. Activity 1 begins with student exploration of the Triangular Inequality form as an Action Step. Furthermore, student activities in working on several questions that refer to the use and understanding of some absolute value properties as a Process Step, thus students are expected to be able to write a formal proof of the triangle inequality as an Object Step. To be able to see whether the scheme for proving the triangle inequality has been formed in the student's thinking process, problem number 1 is given in the Exercise section. Activity 2 is designed to show another way of proving the Triangle Inequality Theorem. Activity 2 begins with the activity of paying attention to and understanding some steps of proving triangle inequality as Action. Followed by the activity of reviewing and determining which absolute value properties are the basis for each step of the proof as a process and writing the complete triangle inequality proof. The scheme formed in the construction of students' thinking framework is implemented in working on exercise number 2.

The validation process of the HLT and LAM design was carried out by discussing it with peers and experts to get input and suggestions for improvement so that it is ready to proceed to the next stage. In Table 1, the comments and suggestions given by the experts are presented.

Table 1. Expert Validation Results

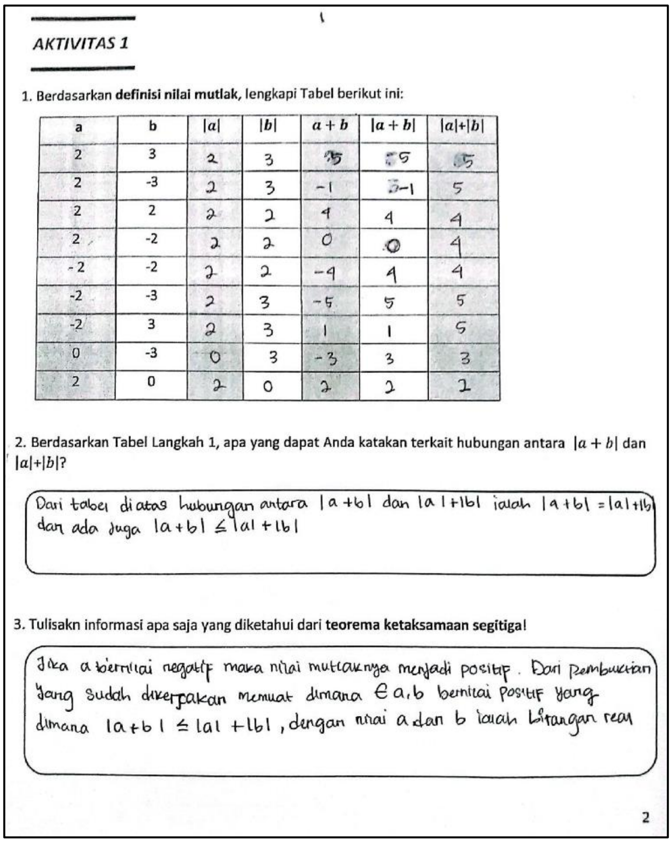
Validator	Comments/Suggestions
1	Student Activity Sheet (LAM) Real Number Absolute Value Material on question number 3 in Activity 2 can be added to the $\|a\|$ and $\|b\|$ columns so that students can understand the material in a structured manner. Recheck the Answer Key to question number 3 in Activity 2, especially the fourth column of row three and row six
2	Use KKO on learning objectives that match the content in the Student Activity Sheet.
3	If necessary, add examples and materials

Valid and practical activity sheets are proven to be able to help and facilitate students in understanding the material, in accordance with the results of research by (Saftari & Sariman, 2021) (Haswati, Bilda, & Nopitasari, 2019) (Ningsih, Darmawijoyo, & Hartono, 2015).

2. Design Experiment

The design experiment was conducted in two phases, namely pilot experiment and teaching experiment. In the pilot experiment, the HLT that had been made was tested in actual learning. The pilot test was conducted on non-subject students. The snippets of students' answers on LAM Activity 1 and Activity 2 are presented in Figure 2 and Figure 3. In Figure 2, it can be seen that the subject has no problems in working on instruction number 1 as the Action Step of the APOS theory. However, the subject seems to have difficulty in understanding and exploring the information in the theorem to be proven, as seen in the subject's answer to question number 3. According to DeVries to develop each concept starts from the individual's mind with an Action. Students at the action level can work through actions, but they do not think as a whole and predict the results (Hartati, 2014). Different results are shown in Septiati's research (2021) that students' ability to construct mathematical proofs is dominant in the indicator of identifying what is the data / information from the statement. The level of difficulty of the material is one of the reasons for this difference.

Based on what the subject did as well as the results of observations, the subject experienced problems in understanding the questions given, especially for questions number 1 and 2. However, after being given guidance that the basis for doing proof is the definition and theorem and its properties, the subject was able to answer the questions given.



AKTIVITAS 1

1. Berdasarkan **definisi nilai mutlak**, lengkapi Tabel berikut ini:

a	b	$\|a\|$	$\|b\|$	$a+b$	$\|a+b\|$	$\|a\|+\|b\|$
2	3	2	3	5	5	5
2	-3	2	3	-1	1	5
2	2	2	2	4	4	4
2	-2	2	2	0	0	4
-2	-2	2	2	-4	4	4
-2	-3	2	3	-5	5	5
-2	3	2	3	1	1	5
0	-3	0	3	-3	3	3
2	0	2	0	2	2	2

2. Berdasarkan Tabel Langkah 1, apa yang dapat Anda katakan terkait hubungan antara $|a+b|$ dan $|a|+|b|$?

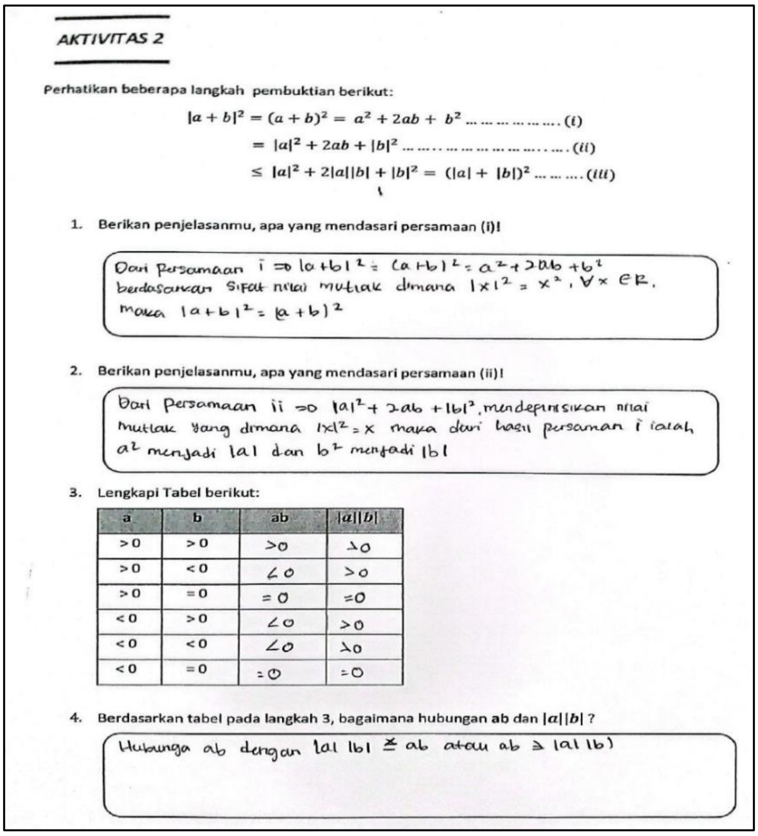
Dari tabel diatas hubungan antara $|a+b|$ dan $|a|+|b|$ ialah $|a+b| = |a|+|b|$ dan ada juga $|a+b| \leq |a|+|b|$

3. Tulisakn informasi apa saja yang diketahui dari **teorema ketaksamaan segitiga**!

Jika a bernilai negatif maka nilai mutlaknya menjadi positif. Dari pembuktian yang sudah diterapkan memuat dimana $\in a, b$ bernilai positif yang dimana $|a+b| \leq |a|+|b|$, dengan nilai a dan b ialah bilangan real

2

Figure 2. Activity 1 Response Capture



AKTIVITAS 2

Perhatikan beberapa langkah pembuktian berikut:

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 \ldots\ldots (i)$$
$$= |a|^2 + 2ab + |b|^2 \ldots\ldots (ii)$$
$$\leq |a|^2 + 2|a||b| + |b|^2 = (|a|+|b|)^2 \ldots\ldots (iii)$$

1. Berikan penjelasanmu, apa yang mendasari persamaan (i)!

Dari persamaan i $\Rightarrow |a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2$ berdasarkan sifat nilai mutlak dimana $|x|^2 = x^2, \forall x \in R$. maka $|a+b|^2 = (a+b)^2$

2. Berikan penjelasanmu, apa yang mendasari persamaan (ii)!

Dari persamaan ii $\Rightarrow |a|^2 + 2ab + |b|^2$, mendefinisikan nilai mutlak yang dimana $|x|^2 = x$ maka dari hasil persaman i ialah a^2 menjadi $|a|$ dan b^2 menjadi $|b|$

3. Lengkapi Tabel berikut:

a	b	ab	$\|a\|\|b\|$
> 0	> 0	> 0	> 0
> 0	< 0	< 0	> 0
> 0	= 0	= 0	= 0
< 0	> 0	< 0	> 0
< 0	< 0	< 0	> 0
< 0	= 0	= 0	= 0

4. Berdasarkan tabel pada langkah 3, bagaimana hubungan ab dan $|a||b|$?

Hubunga ab dengan $|a||b| \geq ab$ atau $ab \geq |a||b|$

Figure 3. Activity 2 response capture

In the Teaching Experiment. The revised HLT at the previous stage will be tested again to students who are the research subjects. According to (Risnanosanti, Prasetiyo, & Syofiana, 2023) to get a qualified HLT, it can be carried out repeatedly in the second stage, namely, teaching experiment. Teaching experiments were carried out in 2 meetings, namely the first meeting discussing Activity 1 on LAM and meeting 2 discussing activity 2 along with exercises. According to (Gravemeijer & Cobb, 2006), the purpose of the teaching experiment is to test and refine the alleged local instructional theory that has been developed and to develop an understanding of how it works. The purpose of the teaching experiment is to collect data to answer the research questions. Thus, in this phase the sequence of activities developed in the pilot experiment phase is implemented in the classroom. After carrying out the Action and Process Steps in Activity 1 and Activity 2, the Objects and Schemes are well formed in students as illustrated in Figure 4 to Figure 6

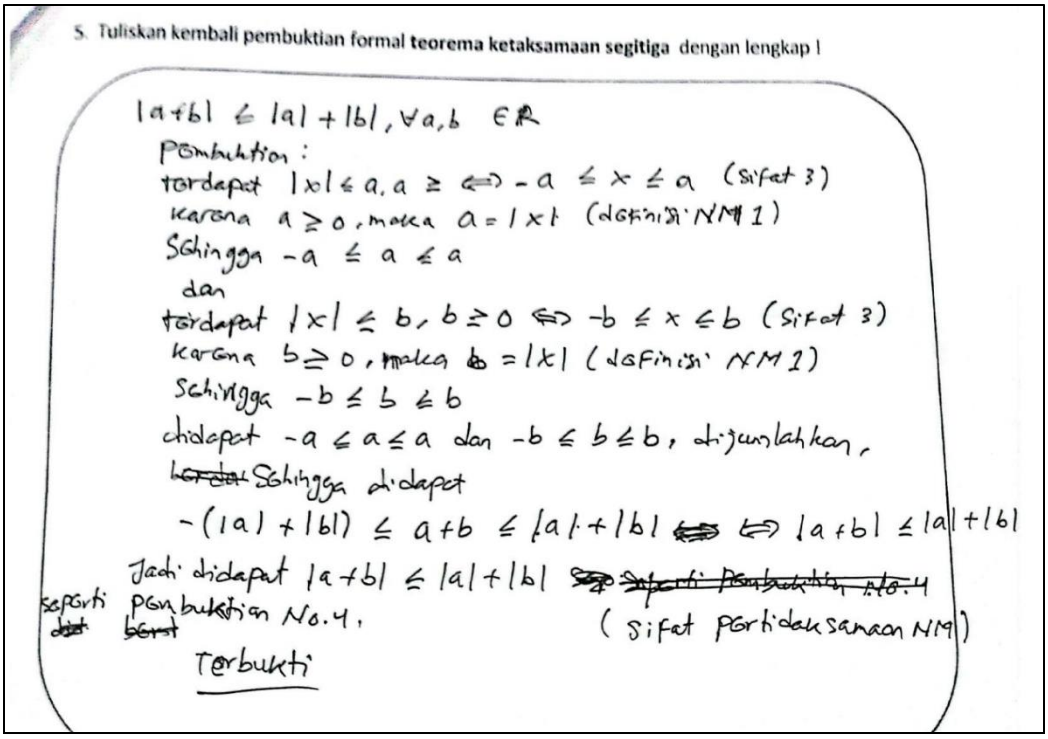
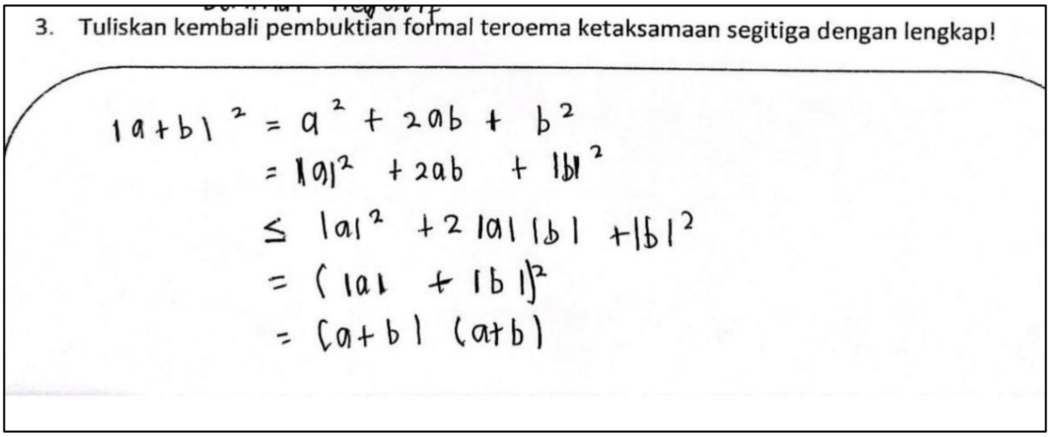


Figure 4: Object in Activity 1

.

In Figure 4, it can be seen that the subject can give the right answer to question number 5 in Activity 1. Question number 5 is an implementation of the Object Step of the APOS theory.

In Figure 5, the subject's answer for the Object Step in Activity 2 is shown. The subject has been able to write the formal proof of the triangle inequality, but it is incomplete. The conception of objects is defined as a form of understanding of a mathematical concept which is the application of actions and processes. Encapsulation of action or process is the ability of individuals to understand action or process as a totality. Individuals become aware that an operation can be performed on the action or process (Hartati, 2014).



3. Tuliskan kembali pembuktian formal teroema ketaksamaan segitiga dengan lengkap!

$|a+b|^2 = a^2 + 2ab + b^2$

$= |a|^2 + 2ab + |b|^2$

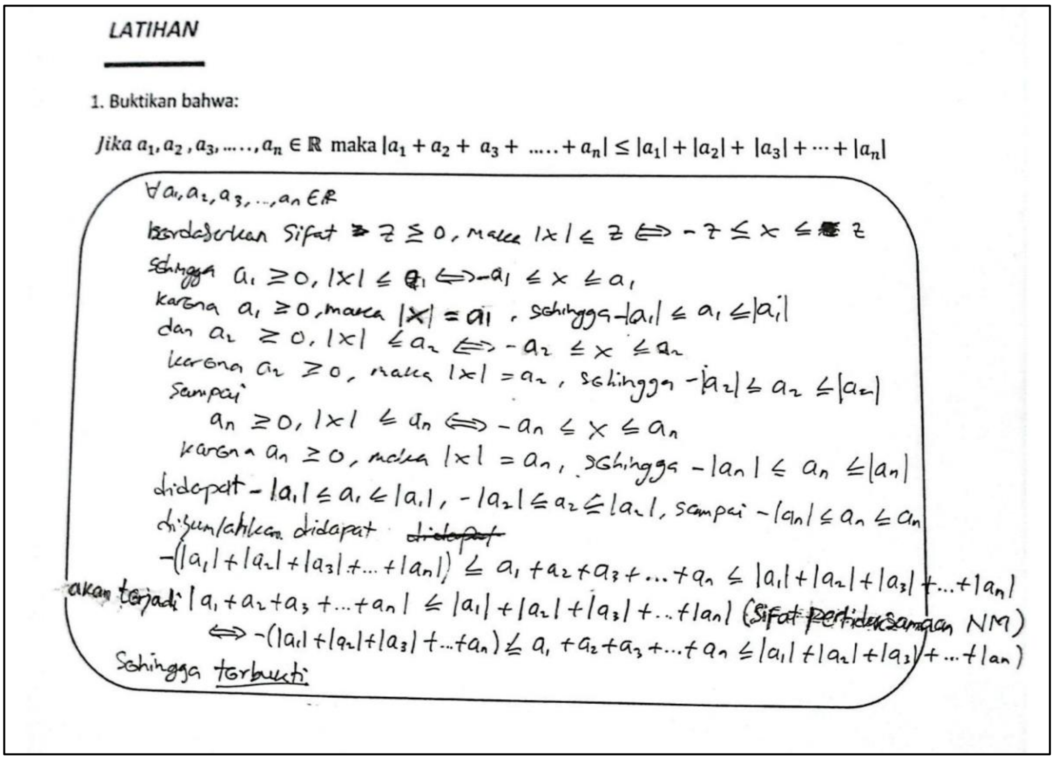
$\leq |a|^2 + 2|a||b| + |b|^2$

$= (|a| + |b|)^2$

$= (a+b| (a+b)$

Figure 5: Object in Activity 2

Based on a series of activities that have been carried out in Activities 1 and 2, the subject is given an exercise to see if a schema for proving triangle inequality has been formed. The answers given by the subject are shown in Figures 6 and 7.



LATIHAN

1. Buktikan bahwa:

Jika $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$ maka $|a_1 + a_2 + a_3 + \ldots + a_n| \leq |a_1| + |a_2| + |a_3| + \cdots + |a_n|$

$\forall a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$

berdasarkan Sifat $z \geq 0$, maka $|x| \leq z \Leftrightarrow -z \leq x \leq z$

Sehingga $a_1 \geq 0, |x| \leq a_1 \Leftrightarrow -a_1 \leq x \leq a_1$

karena $a_1 \geq 0$, maka $|x| = a_1$, Sehingga $-|a_1| \leq a_1 \leq |a_1|$

dan $a_2 \geq 0, |x| \leq a_2 \Leftrightarrow -a_2 \leq x \leq a_2$

karena $a_2 \geq 0$, maka $|x| = a_2$, Sehingga $-|a_2| \leq a_2 \leq |a_2|$

Sampai

$a_n \geq 0, |x| \leq a_n \Leftrightarrow -a_n \leq x \leq a_n$

karena $a_n \geq 0$, maka $|x| = a_n$, Sehingga $-|a_n| \leq a_n \leq |a_n|$

didapat $-|a_1| \leq a_1 \leq |a_1|$, $-|a_2| \leq a_2 \leq |a_2|$, sampai $-|a_n| \leq a_n \leq a_n$

dijumlahkan didapat ~~didapat~~

$-(|a_1| + |a_2| + |a_3| + \ldots + |a_n|) \leq a_1 + a_2 + a_3 + \ldots + a_n \leq |a_1| + |a_2| + |a_3| + \ldots + |a_n|$

akan terjadi $|a_1 + a_2 + a_3 + \ldots + a_n| \leq |a_1| + |a_2| + |a_3| + \ldots + |a_n|$ (Sifat pertidaksamaan NM)

$\Leftrightarrow -(|a_1| + |a_2| + |a_3| + \ldots + a_n) \leq a_1 + a_2 + a_3 + \ldots + a_n \leq |a_1| + |a_2| + |a_3| + \ldots + |a_n|)$

Sehingga terbukti

Figure 6. Exercise Number 1

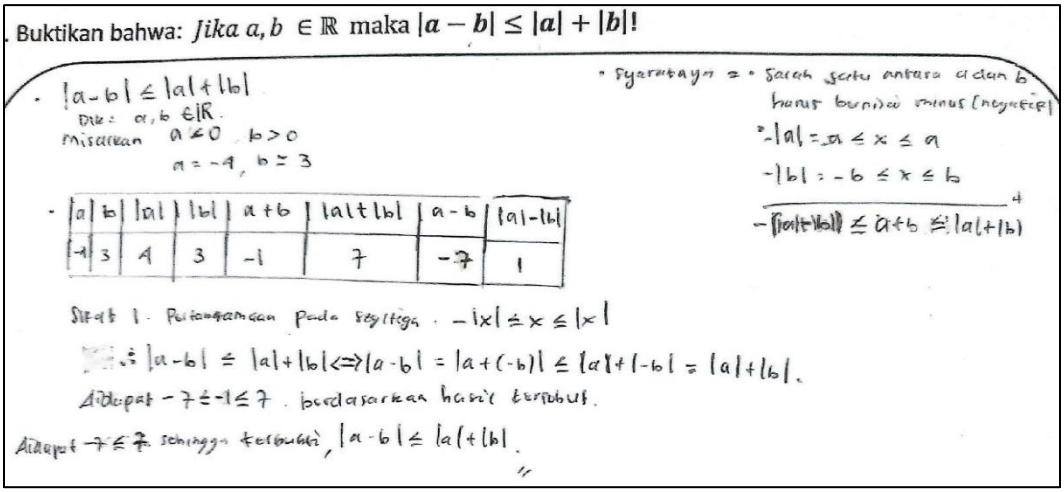


Figure 7. Exercise Number 2

Based on Figures 6 and 7 the subject has given the right answer, meaning that a scheme has been formed in proving the triangle inequality. This is in line with what is expressed by (Wilhelmi, Godino, & Lacasta, 2007) that the introduction of absolute values through arithmetic partial meanings is not representative: any analytical partial meaning cannot be handled with assurance (function theory is beyond students' knowledge); vectorial partial meanings can only be explained in natural language (not formalized); and, finally, geometric partial meanings are understood as a simple rule of "removing the minus sign". However, this was not the case for all subjects. Not all subjects were able to prove the given statement.

3.Retrospective Analysis

At the Retrospective Analysis stage, the data obtained was analyzed by looking at the suitability between the HLT that had been made and the actual conditions during the experiment. The conformity between the HLT conjecture and the ALT (Actual Learning Trajectory) is presented in Table 2.

Table 2. Comparison of HLT Learning Process Conjecture with ALT

HLT	ALT
Students understand the properties of absolute value: - Use the properties of absolute values to do a proof - Determine what properties are used to do the proof.	 - Students are able to use the properties of absolute values to carry out proofs. - Students are able to determine what properties are used to carry out proofs
Students understand formal proof: - write down the complete proof of the triangle inequality theorem	 -students have not written the proof of the triangle inequality theorem completely
Students understand the proof of the Triangle Inequality Theorem: - Doing 2 exercise problems correctly	 - students have not done 2 exercise questions correctly

Based on Table 2, it is known that the Object Step in APOS theory has not been fulfilled properly by the research subject. This has an impact on not forming a scheme on the subject in carrying out the proof. Not all subjects were able to prove the given statement. This condition is the same as what happened in Supriadi's research (2021), that only 3 out of 9 research subjects were able to achieve the scheme step of the APOS theory. (Hartati, 2014) also stated that although students at the action level can do well, they do not think as a whole and predict the results.

According to Reid (2001), proof is basically making a series of deductions from assumptions (premises or axioms) and existing mathematical results (lemmas or theorems) to obtain important results from a mathematical problem. When the subject cannot understand well the existing definitions and theorems, there will be difficulties in carrying out the proof. When the learning that is done is not in accordance with the design that has been designed, it is necessary to redesign (thought experiment) of the HLT and then test the HLT again (instruction experiment). Tracing the learning trajectory is important to support students in developing an understanding of what is being learned (Wijaya, Elmaini, & Doorman, 2021).

CONCLUSION

Based on the results and findings in the study, it can be concluded that the activities designed based on APOS theory cannot fully support evidence-based learning. The series of activities are understanding the definition or theorem as a form of Action, using the properties of absolute values as a form of Process, writing the proof of the theorem as a form of Process and solving problems related to the triangle inequality theorem as a form of Scheme. Thus, it is recommended to revise the learning trajectory that has been designed especially at the object and schema stages of APOS theory and continue the trial as the next cycle.

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